Concepts in Matrix Algebra and Matrix Method

Matrices are everywhere. The basic mathematical operations such as addition, subtraction, multiplication for large number of variables and equations can be performed in a systematic way using matrices.

This module will deal with the different concepts that are used in matrix algebra and discuss the methods that may be used to solved systems of simultaneous equations.

Objectives:

The objectives of the nodule are:

1. *Recognize* the basic concepts underlying matrices
2. *Explore* the geometry behind the concepts used in matrix
3. State the methods to solve simultaneous equations using matrices

Terminology:

1. Linear Transformation- a function from one vector to space that preserves the original structure
2. Linear Combination- linear sum or linear difference of vectors
3. Transpose of a matrix- matrix obtained by interchanging the rows and columns
4. Determinant- volume enclosed by the rows of a matrix
5. Rank of a matrix: the total number of linearly independent rows or columns in a matrix.

12.1. Linear Transformation and Linear Combination

The word “transformation” and “change” have a thin line of distinction. Change may be viewed as something that is different from the original. In other words, it tends to alter the past. But transformation may be thought of as something that can retain some characteristics of the original. In other words, transformation works towards a better future ignoring the past.

In mathematical language, a transformation is a function ‘f’ that maps a set X to itself. Transformation may be diverse in nature; some can be complex and difficult to understand. The concept of linear transformation is generally used in matrix algebra.

A) Graphical representation of change and transformation:

The difference between change and transformation may be explained using the concept of the demand curve from the Theory of Consumer Demand. The most widely used demand function is the linear demand function that results in a straight line demand curve.

There are two important concepts that we are familiar with regard to the demand curve.

a) Movement along the demand curve
b) Shift in the demand curve

Graphically, it may be shown in Fig 1 below. AB is the demand line and quantity demanded from A to B is a movement along the demand curve. This may be seen as a transformation.

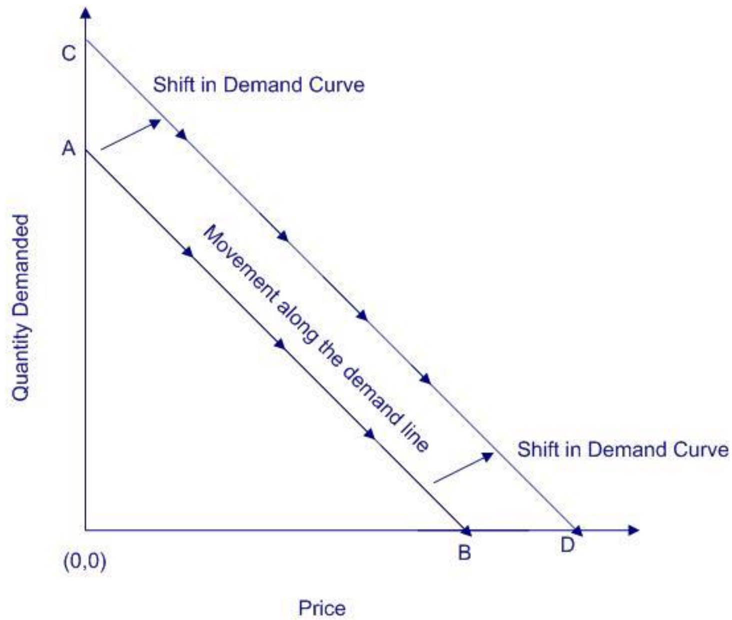

Fig 1: Change and transformation

In Fig 1, a movement along the demand curve shows the different quantities demanded at different prices. However, it retains the original level of all the other factors that may affect demand. (*"ceteris paribus" meaning "other things remaining the same"*).

Since the original curve is not disturbed, this may be seen as a transformation.

On the other hand, a shift in the demand curve occurs due to the change in the external factors. When there are changes in the other factors (such as income of the consumer), the demand curve shifts to a new position. The original demand curved is totally shifted and it does not retain anything from the past demand curve. This may be seen as a Change in the Demand.

B) Characteristics of a linear transformation:

There are two basic characteristics of linear transformation:

a) The transformation will be linear. Linear lines are straight lines. Therefore, if the original line is a straight line, after transformation, the lines remains a straight line. If it becomes a curve, it is not a linear transformation.
b) The point of origin remains undisturbed and movements take place around the origin.
c) The transformed lines must be parallel and evenly spaced.

A linear transformation may be seen as functions that preserves scalar multiplication and addition of vectors.

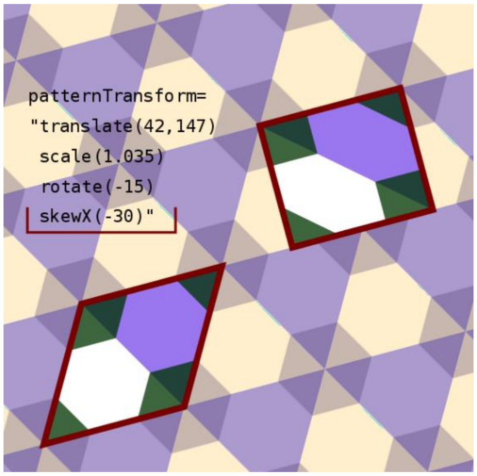

Fig 2: Linear Transformation

[Source:https://en.wikipedia.org/wiki/Transformation_(function)#/media/File:A_code_snippet_for_a_rhombic_repetitive_pattern.svg]

Fig 2 shows a composition of four mappings in SVG (Scalable Vector Graphics), which transforms a rectangular repetitive pattern into a rhombic pattern. The four transformations are linear.

C) Matrix, Linear Transformation and Linear Combination

A linear combination is the linear sum or linear difference of vectors. In simple words, a linear combination of x and y is an expression constructed by multiplying each term by a constant and adding the results, that is, ax +by, where a and b are constants.

If we have two vectors, say $A1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $A2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and we wish to see if the vector $B = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ is a linear combination of the vectors V1 and V2.

Clearly, $2A1 = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $4A2 = 4\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$ (Scalar multiplication)

Now, 2A1 + 4A2 = $\begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = B$ (Vector addition)

Thus we can say that vector B is a linear combination of vector A1 and A2

12.2. Transpose of a matrix

The transpose of a matrix was introduced by Arthur Cayley, a British Mathematician. “Transpose” means “to interchange” or “exchange places” or “transfer”. Following this meaning of transpose, the transpose of a matrix is the matrix obtained after interchanging the rows and columns.

If A is a matrix, its transpose is denoted by A' or A^T. The order of the transpose remains the same as the original matrix.

Thus, if $A = \begin{bmatrix} 2 & 3 & 6 \\ 0 & 1 & 1 \\ 3 & 0 & 2 \end{bmatrix}$ is a 3 x 3 matrix, The transpose of A will be $A' = \begin{bmatrix} 2 & 0 & 3 \\ 3 & 1 & 0 \\ 6 & 1 & 2 \end{bmatrix}$, which is also a 3 x 3 matrix.

A) Significance of transpose of a matrix in Economics:

Imagine a two-commodity economy. Let us name the commodities as Good X and Good Y. Also imagine an economy with only two individuals. Let us name the individuals as A and B. Suppose that A consumes two units of Good X and 5 units of Good Y and B consumes 1 unit of Good X and 2 units of Good Y in a day.

To keep a record of this information, we may take the help pf matrix and represent it as:

$$R = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}$$

The first row represents the units of Good X consumed by A and B the second row represents the units of Good Y consumed by A and B.

Suppose the price of Good X is ₹5 and the price of Good Y is ₹2. In matrix form, this may be written as a price vector as

$$P = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

If we wish to calculate the expenditure of A and B, simple algebraic calculation gives,

Expenditure of A = (₹5 x 2) + (₹2 x 5) or, ₹20

Expenditure of B = (₹5 x 1) + (₹2 x 2) or, ₹9

If the matrix R is multiplied by vector P, it gives

$$RP = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$\text{Or, } RP = \begin{bmatrix} (2\ x\ 5) + (1\ x\ 2) \\ (5\ x\ 5) + (2\ x\ 2) \end{bmatrix}$$

$$\text{Or, } RP = \begin{bmatrix} 12 \\ 29 \end{bmatrix}$$

Economically, this multiplication does not give the expenditure of A and B

Now, if we take the transpose of R and multiply it with vector P, it gives,

$$R'P = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$\text{Or, } R'P = \begin{bmatrix} (2\ x\ 5) + (5\ x\ 2) \\ (1\ x\ 5) + (2\ x\ 2) \end{bmatrix}$$

$$\text{Or, } \begin{bmatrix} 20 \\ 9 \end{bmatrix}$$

This vector gives exactly the values of the expenditure incurred by A and B

12.3. Determinant

The term "determinant" means something that determines. In mathematical language, determinant is a term used in Matrix Algebra and is used to denote the volume enclosed by the rows of a matrix. The use of the term "determinant" and "resultant" is attributed to Seki Takakazu, a Japanese mathematician who has been described as Japan's Newton.

Seki Takakazu

[Source: https://en.wikipedia.org/wiki/Seki_Takakazu#/media/File:Seki.jpeg]

The determinant of a matrix, say, $M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is denoted by$M = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ or $\det(M) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$

A) Graphical representation of determinants:

Let us try to graphically find the determinant, or find the volume enclosed by a matrix.

Example,

The matrix M is a collection of two row vectors, $[1 \quad 0]$ and$[0 \quad 1]$.

Graphically, these vectors may be denoted by the arrows with co-ordinates in the graph below.

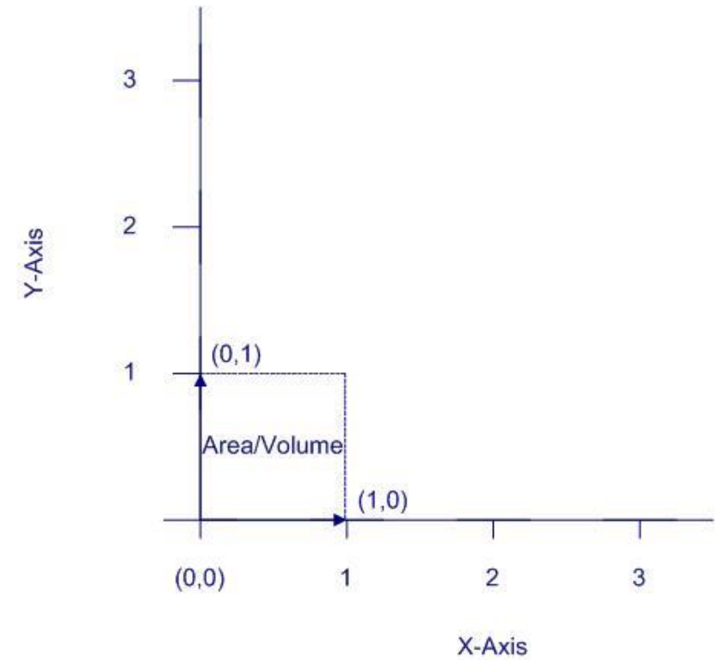

Fig 3: Graphical representation of Determinant

The area or volume enclosed by the two vectors, (0,1) and (1,0) may graphically be calculated by drawing two lines, one parallel to the x-axis from point 1 on the y-axis and another parallel to the y-axis from point 1 on the x-axis. The area or volume is calculated by using the formula for the area of a square. This gives the area as 1.

B) Calculating the Determinant mathematically

For the same matrix, $M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the determinant is calculated by cross multiplication as follows:

Thus, $M = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ = (1x1) – (0x0) = 1

Notes:

1. Determinant is a scalar quantity.

2. Determinants can be calculated only for square matrices.

12.4. Rank of a matrix

The Rank of a matrix is the dimension of vector space (also called linear space) generated. The Rank of a matrix may be defined as the total number of linearly independent rows or columns in a matrix. In other words, "rank" is the dimension of the output in a linear transformation. For example, when the output of a linear transformation is two dimensional, it has a rank 2.

A) Linear Dependence and linear Independence

As mentioned earlier, matrix is a collection of vectors. These vectors within an array may be dependent or independent. If the vectors are dependent on some other vector, they are said to be linearly dependent, otherwise they are linearly independent. A simple case of linear dependency is when a vector is a multiple of some other vector.

Example,

The two column vectors $C_1 = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$ and $C_2 = \begin{bmatrix} 25 \\ 50 \end{bmatrix}$ are linearly dependent because C_2 is five times of C_1 Or, $5\begin{bmatrix} 5 \\ 10 \end{bmatrix} = \begin{bmatrix} 25 \\ 50 \end{bmatrix}$

B) Calculating rank of a matrix:

Rank may be defined as the maximum order of a non-zero determinant that can be drawn from the rows and columns of that matrix

The Rank of a matrix may be calculated by finding a non-zero determinant. As stated earlier, a determinant can be calculated only for a square matrix, where the number of rows and columns are equal. Now, if there is a rectangular matrix of order say, 3 x 2, (number or rows is not equal to number of columns), the maximum order for which a determinant may be constructed is 2 x 2. If any determinant of order 2 x 2 is non-zero, the rank of the matrix of order 3 x 2 will be 2.

Example,

Let $V = \begin{bmatrix} 3 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}$ be a matrix of order 3 x 2 (3 rows and 2 columns)

The possible determinants of order 2 x 2 may be, $V1 = \begin{vmatrix} 3 & 1 \\ 2 & 0 \end{vmatrix}$, $V2 = \begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix}$ and $V3 = \begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix}$

det(V1) = -2, det(V2) = -1 and det (V3) = 0

Since there exists at least one non-zero determinant of order 2 x 2, therefore, it can be said that the rank of the matrix V is 2. To put it in other words, it may be said that there are two linearly independent rows in the matrix.

(By observation, you may find that the second row is linearly dependent as, adding 1 to the elements of second row gives the first row.)

12.5. Matrix Methods:

Matrix Methods are the methods that are used to solve any system of linear equations using matrix algebra. The two most popularly used methods are:

1. Matrix Inverse and
2. Cramer's Rule

A) Matrix Inverse

"Inverse" means reciprocal. For a scalar quantity, X, the reciprocal of X may be denoted by 1/X. In case of a matrix, the reciprocal of a matrix is denoted by X^{-1}.

The matrix conversion for the linear equation model,

$$4x + 2y - z = 40$$

$$2x + 3y \qquad = 43$$

$$x \qquad + z = 38$$

may be written as

$$\begin{bmatrix} 4 & 2 & -1 \\ 2 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 40 \\ 43 \\ 38 \end{bmatrix}$$

Where $E = \begin{bmatrix} 4 & 2 & -1 \\ 2 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ is the co-efficient matrix, $R = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is the matrix of variables and

$C = \begin{bmatrix} 40 \\ 43 \\ 38 \end{bmatrix}$ is the constant matrix.

In short hand, ER = C

Now, the task is to find the values of the elements in the matrix R, that is find the values of x, y and z.

Thus, using simple simplification rule, we may write, $R = E^{-1}C$

To find the values of the elements in the matrix R, we must find E^{-1}, or what is known as inverse of E. By multiplying with vector C, we arrive at the vector R.

Since this method of solving involves the inverse, it is known as the Matrix Inverse Method.

B) Cramer's Rule:

Cramer's rule is named after Gabriel Cramer. In this method, instead of matrix inverse, determinants are used. The rule for solving a system of linear equations by the Cramer's rule may be explained as follows:

1. If E = $\begin{bmatrix} 4 & 2 & -1 \\ 2 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ is the co-efficient matrix, we require four determinants

a) |E| = $\begin{vmatrix} 4 & 2 & -1 \\ 2 & 3 & 0 \\ 1 & 0 & 1 \end{vmatrix}$

b) $|E_x|$ = $\begin{vmatrix} 40 & 2 & -1 \\ 43 & 3 & 0 \\ 38 & 0 & 1 \end{vmatrix}$ is the determinant obtained by replacing the first column of the

co-efficient matrix by the elements of the constant vector.

c) $|E_y|$ = $\begin{vmatrix} 4 & 40 & -1 \\ 2 & 43 & 0 \\ 1 & 38 & 1 \end{vmatrix}$ is the determinant obtained by replacing the second column of the

co-efficient matrix by the elements of the constant vector

d) $|E_z|$ = $\begin{vmatrix} 4 & 2 & 40 \\ 2 & 3 & 43 \\ 1 & 0 & 38 \end{vmatrix}$ is the determinant obtained by replacing the third column of the

co-efficient matrix by the elements of the constant vector.

Thus, $x = \frac{|Ex|}{|E|}$ or $x = \frac{\begin{vmatrix} 40 & 2 & -1 \\ 43 & 3 & 0 \\ 38 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 4 & 2 & -1 \\ 2 & 3 & 0 \\ 1 & 0 & 1 \end{vmatrix}} = \mathbf{\frac{148}{11}}$

Similarly, $y = \frac{|Ey|}{|E|}$ or $y = \frac{\begin{vmatrix} 4 & 40 & -1 \\ 2 & 43 & 0 \\ 1 & 38 & 1 \end{vmatrix}}{\begin{vmatrix} 4 & 2 & -1 \\ 2 & 3 & 0 \\ 1 & 0 & 1 \end{vmatrix}} = \mathbf{\frac{59}{11}}$

And $Z = \frac{|Ez|}{|E|}$ or $Z = \frac{\begin{vmatrix} 4 & 2 & 40 \\ 2 & 3 & 43 \\ 1 & 0 & 38 \end{vmatrix}}{\begin{vmatrix} 4 & 2 & -1 \\ 2 & 3 & 0 \\ 1 & 0 & 1 \end{vmatrix}} = \mathbf{\frac{270}{11}}$

Since the determinant is a scalar quantity, the values of x, y and z will be scalar quantities.