Mathematical Economics

Differential Equations: An Introduction

29.1 Introduction:

Change is an inevitable process. There are a large number of variables that increase or decrease, grow or decline with time. For example, time path of price change, growth of population, epidemic growth of a disease, depletion of ozone layer, growth of financial market and so on. These changes cannot be measured by simple algebraic equations.

This module will explore the static and dynamic nature of economic variables and introduce the concept of differential equations.

Objectives

The objectives of this module are:

1. *Differentiate* between economic statics and economic dynamics
2. *Explain the concept and structure of a differential equation*

Terminology

1. Economic statics: a situation where economic variables are studied at one point of time.
2. Micro-Statics: deal with static situations involving microeconomic variables.
3. Microeconomic variable: variable that is studied in microeconomics; individual indicator
4. Macro-Statics: deal with static situations involving macroeconomic variables.
5. Macroeconomic variable: variable that is studied in macroeconomics; indicator that studies the whole economy
6. Economic dynamics: a study of the path of change of economic variables from one equilibrium point to another
7. Micro Dynamics: the dynamics of microeconomic variables such as demand, supply and price, over periods of time
8. Macro Dynamics: the dynamics analyzing economic growth, trade cycles and inflation.
9. Stock variable: a variable measured at a particular point of time that may have accumulated from the past. Example: money or wealth
10. Flow variable: A flow is measured per unit of time. Example: Income, savings

11. Discrete variable: A discrete variable is a variable that can take distinct values or countable values.
12. Continuous variable: A continuous variable is a variable that can take infinite values.
13. Differential equations: equations involving derivatives

29.2 Economic Statics and Economic Dynamics

A. Economic Statics:

The word "static" comes from the Latin word "staticus" and Greek word "staticos" which means "causing to stand". In Mechanics, "static" is used to mean something at rest or that which is without movement. Economic Statics has the following characteristics:

a) **A state of equilibrium**- A system is said to be in equilibrium when the variables are at rest. Economic Statics may be considered as a situation where economic agents are simultaneously satisfied and economic variables are equal to each other.

b) **Economic variables are assumed to be constant**- The concept of Economic Statics is important primarily to understand the complex economy with ease by assuming that certain variables are kept constant.

c) **Economic variables are studied at one point of time**- in such a situation the economy behaves such that the output produced is consumed in the same period and there is no time lag.

Economic Statics may be divided into Micro-Statics and Macro-Statics.

i) Micro-Statics

Micro-Statics deal with situations involving microeconomic variables. For example, price is an important determinant of quantity demanded and supplied and all other factors such as income, tastes, preferences, technology, etc. remain the same, while determining the market price. The behavior of price in market is understood only under the given condition of market equilibrium, that is the condition that demand is equal to supply, or buyers and sellers are satisfied or their tastes and preferences do not change.

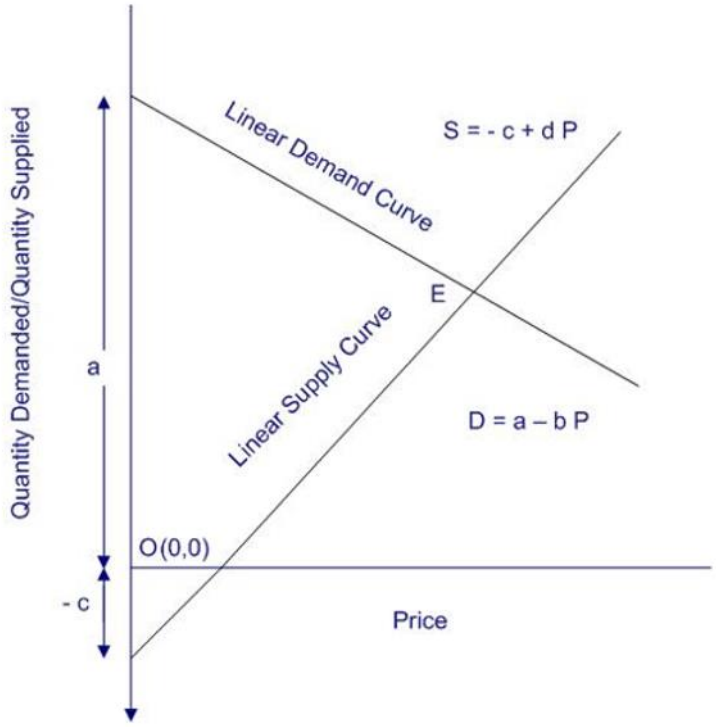

Fig. 29.1: An Example of Micro-Statics

ii) Macro-Statics

Macro-Statics deal with situations involving macroeconomic variables. For example, national income is determined by the intersection of aggregate demand and aggregate supply. The determination of national income may be understood with the help of the static Keynesian Model as shown in Fig. 29.2.

The 45° line is the aggregate supply and the line C + I is the aggregate demand. The intersection of aggregate demand and aggregate supply at point E gives the equilibrium of the economy that determines the national income.

This theory is an example of macro-statics as it shows a timeless equilibrium because it does not consider the time lags involved between the receipt of income and expenditure on consumption goods and in producing consumption goods.

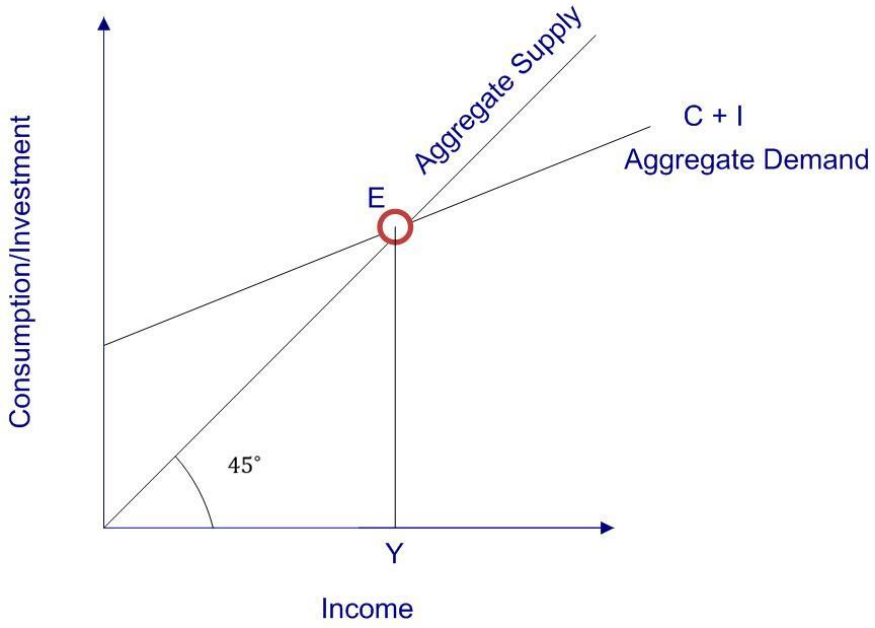

Fig. 29.2 An Example of Macro Statics

B. Economic Dynamics:

The word "dynamic" comes from the Greek word "dunamis", meaning "power". In Mechanics, "dynamics" is concerned with the study of forces and their effects on motion. In mathematics, a dynamical system is a system in which a function depends on time in a geometrical space. The rule of change of a dynamical system is that the future state follows from a current state. Apart from physics and mathematics, a dynamical system has its applications in other fields such as biology, chemistry, engineering, economics and medicine.

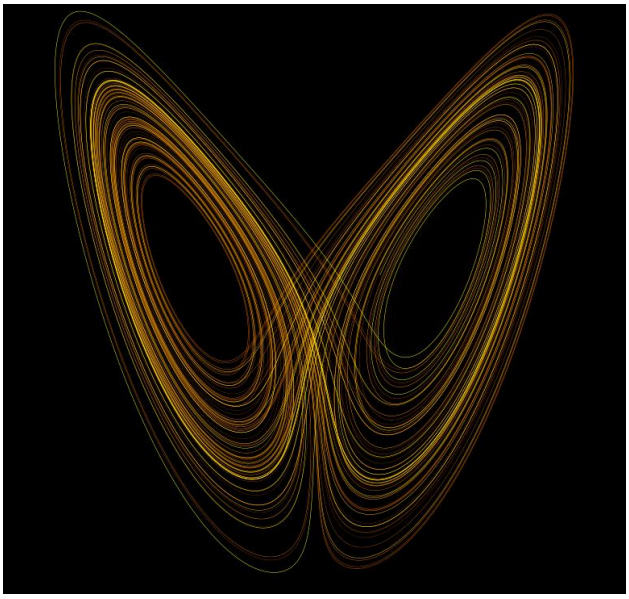

Image 29.1: The Butterfly Effect

[Source: https://en.wikipedia.org/wiki/File:Lorenz_attractor_yb.svg]

The "butterfly effect" describes how a small change in the current state of a deterministic non-linear system results in large differences in future state. This means that the current state depends on the initial state.

Economic dynamics has the following characteristics:

a) Change in equilibrium- dynamic analysis examines if an equilibrium is stable or convergent, unstable or divergent or constant. An equilibrium is said to be stable or convergent if there is a tendency to go back to original equilibrium position after external disturbances. An equilibrium is said to be unstable or divergent if there is tendency to move away from the original equilibrium. An equilibrium is said to be constant if the variables change with constant amplitude around the equilibrium point.

b) **Relationship of economic variables over time-** dynamic analysis tries to discover the functional relationships of economic variables at different points of time.

c) **Time is an important element**- Economic Dynamics studies the path of change of economic variables from one equilibrium point to another. It is concerned with time lags, growth, rates of change, past and expected values of the variables and economy takes time to adjust. In dynamic analysis, current values of economic variables depend on the past values of its own or on other variables.

Economic Dynamics may be divided into micro dynamics and macro dynamics.

i) Micro Dynamics:

The dynamics of microeconomic variables such as demand, supply and price, over periods of time is the main concern of micro dynamics. The Cobweb Model is an example of micro dynamics that explains periodic fluctuation in prices. Nicholas Kaldor analyzed the model in 1934, coining the term "Cobweb Theorem". The Cobweb Model may display convergent, divergent or constant movement.

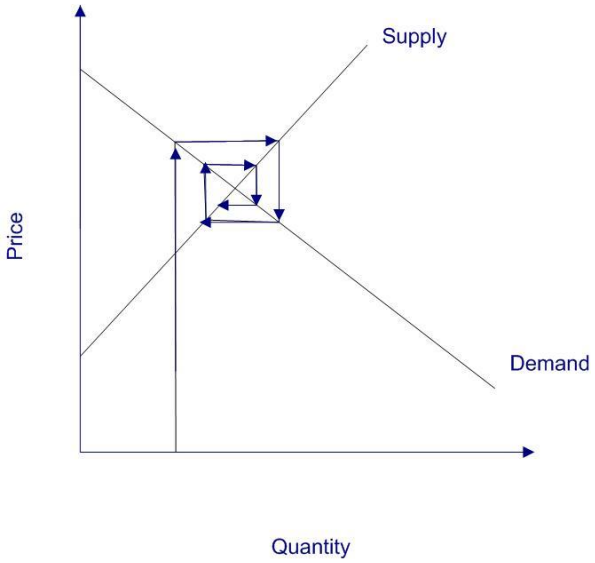

Fig. 29.3 Convergent Movement

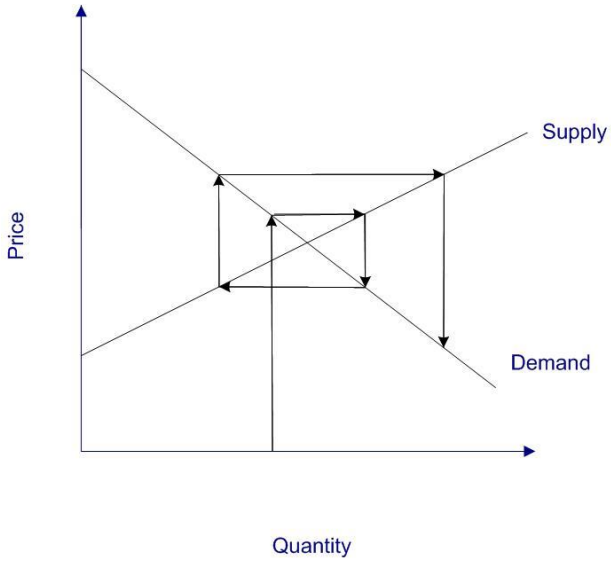

Fig. 29.4 Divergent Movement

In Fig. 29.3 the arrows are seen moving towards the equilibrium point (point of intersection of demand and supply curves) In Fig. 29.4, the arrows are seen to move away from the equilibrium point.

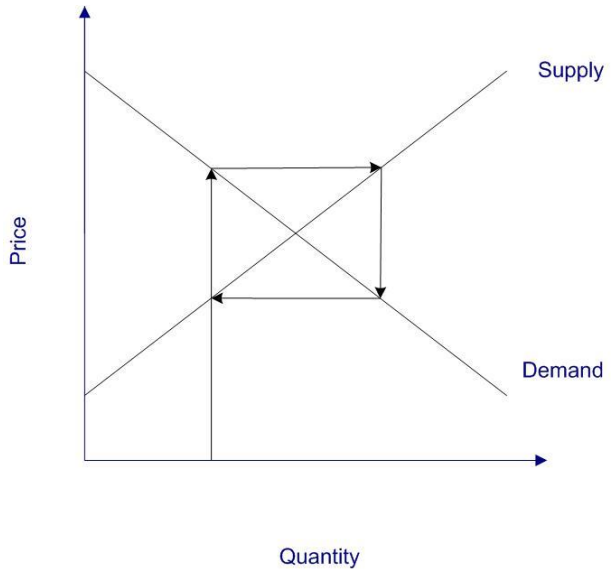

Fig. 29.5 Constant Movement

In Fig. 29.5, the arrows are seen to rotate around a constant path. It neither moves toward or away from the equilibrium.

ii) Macro Dynamics:

Macro Dynamics deals with macroeconomic variables. It is particularly helpful in analyzing economic growth, trade cycles and inflation. The investment multiplier is an example of macro dynamics. It relates to the time lags in the process of income generation that is the gap between the change in income and change in consumption/investment

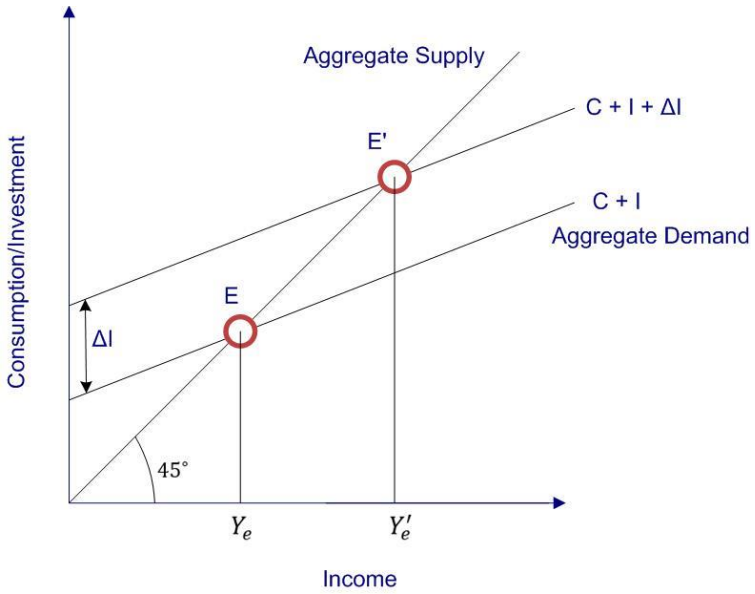

Fig. 29.6 Dynamic Multiplier

29.3 Stock and Flow variables

Stock variables and flow variables are important concepts used in Economics, Business and other related fields. A stock variable is measured at a particular point of time that may have accumulated from the past. A flow is measured per unit of time.

Example:

Money or wealth is a stock but income or savings is a flow. If money is measured in Rupees (₹), income is measured in Rupees (₹) per unit of Time(t), or $\frac{Rupees}{Time}$.

29.4 Discrete and continuous variables

Discrete means distinct or separate. A discrete variable is a variable that can take distinct values or countable values. Continuous means unbroken. A continuous variable is a variable that can take infinite values.

Note: Discrete variables can also take infinite values. Integers are infinite. Real numbers are infinitely dense and infinitely extensive. Discrete numbers are infinitely extensive only.

Example:

The number of items purchased from a market is a discrete variable but the time spent in the market is a continuous variable, because time can be measured infinitely, in fraction of seconds, milliseconds, or nanoseconds and so on.

29.5 Time as a variable:

In Economic Dynamics, time plays a crucial role, because dynamics study how variables behave over a period of time. Generally, it is observed that time behaves as an independent variable and hence is measured on the x axis. The variable whose behavior is to be studied is measured along the y axis.

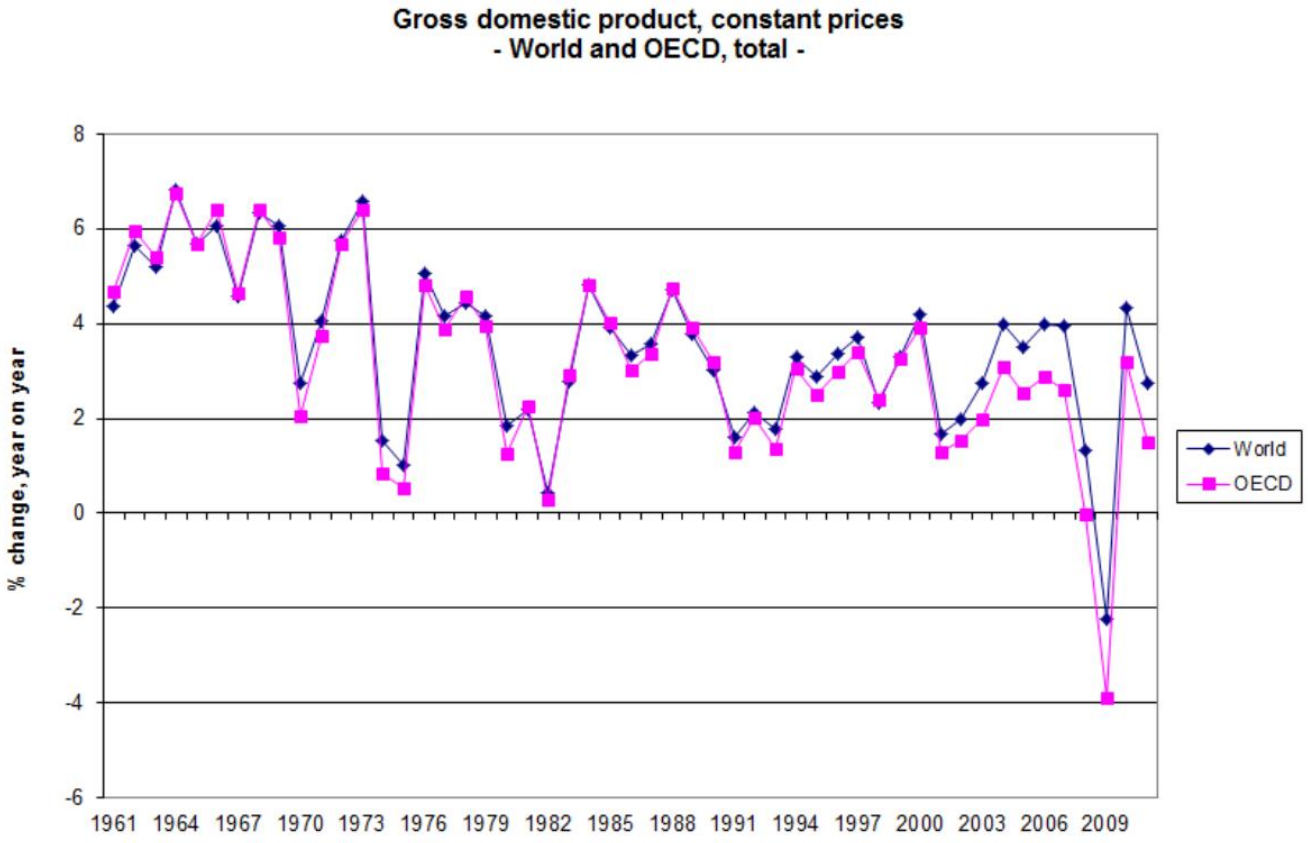

Image 29.1 Rate of change of Gross Domestic Product of the World and Organisation for Economic Co-operation and Development, since 1961

[Source: https://en.wikipedia.org/wiki/File:WeltBIPWorldgroupOECDengl.PNG]

Time may be considered as a continuous variable or as a discrete variable. When time is used as a continuous variable, it is measured in infinitesimally small units, that is there are infinite points between two different points of time. When time is used as a discrete variable, it is measured at distinct points.

29.6 History and Meaning of Differential equations

The concept of differential equation is known since the invention of calculus, historically known as infinitesimal calculus. It involves the study of continuous change. This concept was introduced in social sciences, with neoclassical economics. Neoclassical economics, together with Keynesian economics dominates mainstream economics of today.

Foundations of Economic Analysis, a book by Paul A. Samuelson, is known for formalizing qualitative and quantitative versions of the "comparative statics". “Comparative Statics” is a method for comparing two equilibrium states. It states how the equilibrium changes when parameters are changed.

Image 29.2 Paul A. Samuelson

[Source: https://en.wikipedia.org/wiki/Paul_Samuelson#/media/File:Paul_Samuelson.jpg **]**

The concept of derivative and the process of differentiation are used to determine the rates of change of variables; and equations involving derivatives are known as differential equations. Differential equations therefore may be used to study the behavior of complex economic systems.

Consider the usual function,

$$y = f(x)$$

The derivative is given by $\frac{dy}{dx}$ or $f'(x)$

The general form of a differential equation is written as:

$$\frac{dy}{dx} = f(x)$$

The difference with a simple equation is that, instead of a dependent variable y, a differential equation has the derivative $\frac{dy}{dx}$.

Some of the economic models that use differential equations are listed in Table 29.1

Table 29.1 Economic Models involving Differential Equations

Name of Model	Objective	Differential Equation
Harrod-Domar Growth Model	Explains economy's growth rate by in terms of levels of savings and productivity of capital	$\frac{\dot{Y}}{Y} = sc - \delta$ $\frac{\dot{Y}}{Y}$ *is output growth rate* s *is savings rate* c *is marginal product of capital* δ *is depreciation rate*
Solow-Swan Model	Explains long-run economic growth by considering capital accumulation, population growth, and technological progress	$\dot{k} = sk(t)^{\alpha} - (n + g + \delta)k(t)$ $\dot{k}$ *is the capital intensity over time*
Black-Scholes Model	Model for dynamics of a financial market containing derivative investment instruments	$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$ V *is price of the option as a function of stock price* S *and time* t, r *is risk − free interest rate and* σ *is volatility of the stock*
Malthusian Growth Model	Model of exponential growth regarded in the field of population ecology as the first principle of population dynamics	$\frac{dP}{dt} = rP$ P *is population size*, r *is population growth rate* t *is time*

